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Critical Points of Two-Dimensional Bootstrap Percolation-Like Cellular Automata

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Cellular automata in two dimensions that generalize the bootstrap percolation dynamics are considered, focusing on the threshold p_c of the initial density for convergence to total occupancy to occur; these models are classified according to p_c being 0, 1, or strictly between these extreme values. Explicit upper and lower bounds are provided in the third case.

KEY WORDS: Cellular automata; bootstrap percolation; critical points.

In this paper I consider further the cellular automata studied in ref. 4 in the two-dimensional case. The models are defined as follows. Each site x of $\mathbb{Z}^2$ is at time $t = 0, 1, 2,\ldots$ in state $\eta_t(x) \in \{0, 1\}$. At $t = 0$ the sites are independently in state 1 (occupied) with probability p or in state 0 (vacant) with probability $1 - p$. The evolution in time is deterministic, with the following features. Let $N = (0, 1)$, $W = (-1, 0)$, $S = (0, -1)$, and $E = (1, 0)$ be the nearest neighbors of the origin of $\mathbb{Z}^2$. Set also $\mathcal{N} = \{N, W, S, E\}$. To define the dynamics, take a set $\mathcal{D}$ of subsets of $\mathcal{N}$ with the property of being increasing, i.e., if $A \subset B$ and $A \in \mathcal{D}$, then $B \in \mathcal{D}$. Let the translation by x of a set $A \subset \mathbb{Z}^2$ be written as

$$A + x = \{y \in \mathbb{Z}^2 \colon y = z + x \text{ for some } z \in A\}$$

The evolution is given by the following rules:

(i) If $\eta_t(x) = 1$, then $\eta_{t+1}(x) = 1$.

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(ii) If $\eta_t(x)=0$ and $\eta_t(y)=1$ for all $y \in A+x$ for some $A \in \mathscr{D}$, then $\eta_{t+1}(x)=1$.

(iii) Otherwise $\eta_{t+1}(x)=0$.

I denote by $P_p(\cdot)$ the law of the process starting from density p.

Bootstrap percolation describes the models for which a set is or is not in $\mathscr{D}$ depending only on its cardinality, that is, the cases of the sort

$$\mathscr{D}=\{A \subset \mathscr{N}: |A| \geqslant l\}$$

for some fixed $l=1, 2, 3, 4$.

Adler and Aharony[1] introduced models that they call diffusion percolation, which are closely related to the models that I am considering.

For each one of these models defined above, the density at time t,

$$\rho_t(p)=P_p(\eta_t(0)=1)$$

is nondecreasing in t and hence converges to the asymptotic density

$$\rho_\infty(p)=\lim_{t \to \infty} \rho_t(p)$$

When $\rho_\infty(p)=1$ one says that the process converges to total occupancy. Due to monotonicity in p one naturally defines the critical point

$$p_c=\inf\{p \geqslant 0: \rho_\infty(p)=1\}$$

In this paper I will classify all these two-dimensional models according to p_c being 0, 1, or strictly between these two extreme values. Some explicit upper and lower bounds will also be provided in the third case.

First I recall results from ref. 4. The four oriented models are those with $\mathscr{D}=\{A \subset \mathscr{N}: \mathscr{S} \subset A\}$, where $\mathscr{S}$ is, respectively, $\{E, N\}$, $\{N, W\}$, $\{W, S\}$, or $\{S, E\}$. I (arbitrarily) call the first one the basic oriented model. In Section 4 of ref. 4 it was proven that for the oriented models

$$p_c=1-p^*$$

where p^* is the critical point for oriented site percolation on $\mathbb{Z}^2$. I say that a model I dominates a model II if (with a self-explanatory notation) $\mathscr{D}_{\mathrm{II}} \subset \mathscr{D}_{\mathrm{I}}$, i.e., if it is easier for a vacant site to become occupied in I compared to II. In Section 5 of ref. 4 a simple argument was given showing that if a model does not dominate any oriented model, then this model has $p_c=1$. In other words, if no one of the sets $\{E, N\}$, $\{N, W\}$, $\{W, S\}$, and $\{S, E\}$ is in $\mathscr{D}$, then $p_c=1$. (In our case it is enough to observe that squares of side 2 of vacant sites at time 0 will remain vacant forever.)

Now I consider models that dominate one of the oriented models, and with no loss of generality (due to symmetry), one can suppose that the dominated model is the basic oriented model, i.e.,

$$\{E, N\} \in \mathscr{D} \tag{1}$$

There are two cases:

Case I. Relation (1) holds and at least one of the following sets is also in $\mathscr{D}$: $\{N, W\}$, $\{S, E\}$.

Case II. Relation (1) holds, but no one of the sets $\{N, W\}$, $\{S, E\}$ is in $\mathscr{D}$.

Theorem.

(i) In Case I, $p_c = 0$.

(ii) In Case II, $1-(p^*)^{1/4} \leqslant p_c \leqslant 1-p^*$; in particular, $p_c \in (0, 1)$.

Proof. (i) By symmetry, there is no loss in generality in assuming that $\{\{E, N\}, \{S, E\}\} \subset \mathscr{D}$.

Define the subsets of $\mathbb{Z}^2$:

$$V(i; j) = \{(x, y) \in \mathbb{Z}^2 \colon x = i, 0 \leqslant y \leqslant j\}$$

$$H(i_1, i_2; j) = \{(x, y) \in \mathbb{Z}^2 \colon i_1 \leqslant x \leqslant i_2, y = j\}$$

Consider now the numerical sequence $(m_k)_{k=1,2,\ldots}$ given by

$$m_1 = -1$$

$$m_{k+1} = m_k - k$$

And for $k = 1, 2,\ldots$ define the events

$$F_k = \{\text{For all } i \in \{m_{k+1}+1,\ldots, m_k\} \text{ there is at least one site } x \in V(i; k-1) \text{ such that } \eta_0(x) = 1\}$$

$$G_k = \{\text{For some } x \in H(m_{k+1}+1, m_k; k), \eta_0(x) = 1\}$$

If F_1 and G_1 happen, then the region $V(m_2+1, 1)$ is completely occupied. If also F_2 and G_2 happen, then it is easy to see that $V(m_3+1, 2)$ and $H(m_3+1, m_3; 0)$ will eventually become completely occupied. Reasoning inductively, we see that if all the events F_k, G_k, $k = 1, 2,\ldots$, happen, then in particular all sites of the form $(-h, 0)$, $h = 1, 2,\ldots$, will become eventually occupied. The probability that all these events happen is

$$\alpha(p) = \prod_{k=1}^{\infty} [P(F_k) \cdot P(G_k)] = \prod_{k=1}^{\infty} [1-(1-p)^k]^{k+1}$$

Hence, for every $p > 0$,

$$\alpha(p) > 0 \tag{2}$$

The origin is said to be a good site if all the events F_k, G_k, $k = 1, 2, \ldots$, happen. More generally, one says that the site $x \in \mathbb{Z}^2$ is a good site if all these events happen when η_0 is replaced by the shifted configuration $\theta_{-x}\eta_0$ given by

$$(\theta_{-x}\eta_0)(y) = \eta_0(y + x)$$

By ergodicity and (2), there is with probability one good sites on the x axis to the right of the origin. Therefore, by the remarks above, the origin will almost surely become eventually occupied and

$$\rho_\infty(p) = P_p(\eta_t(0) = 1 \text{ for some } t \geqslant 0) = 1$$

This completes the proof of (i).

(ii) Consider the squares of side 2,

$$Q(x, y) = \{(2x, 2y), (2x+1, 2y), (2x+1, 2y+1), (2x, 2y+1)\}$$

Think of a new lattice $\mathbb{Z}^2$ and declare the site (x, y) of this new lattice to be vacant at time t if and only if the four sites in $Q(x, y)$ are vacant at time t in the original lattice.

One says that double-oriented percolation of vacant sites occurs in the configuration ζ if there is a doubly infinite chain of sites..., z_{-1}, z_0, z_1,... such that $z_0 = 0$, $\zeta(z_i) = 0$, and $z_{i+1} \in \{z_i + (1, 0), z_i + (0, 1)\}$, for $i \in \mathbb{Z}$.

Suppose that at $t = 0$ double-oriented percolation of vacant sites occurs in the new lattice. I will argue that then the same will occur at any later time and in particular $\eta_t(0) = 0$ for every $t \geqslant 0$. Indeed, if $z = (x, y)$ belongs to the doubly infinite chain of vacant new sites which is present at $t = 0$, then in the original lattice the four sites of $Q(z)$ are vacant and:

(a) $(2x, 2y)$ and $(2x+1, 2y+1)$ have at most one occupied neighbor.

(b) $(2x, 2y+1)$ has at most the sites to its north and west occupied.

(c) $(2x+1, 2y)$ has at most the sites to its south and east occupied.

Hence, by the hypothesis, $Q(z)$ will still be completely vacant at time $t = 1$. (Observe that if $\mathscr{D}$ contains a singleton, then it must contain one of the sets

$\{N, W\}$, or $\{S, E\}$.) Induction on t completes now the argument, which leads to

$$1 - \rho_\infty(p) \geqslant P_p(\text{at } t = 0 \text{ there is double-oriented percolation of vacant sites in the new lattice}) \quad (3)$$

But if $(1-p)^4 > p^*$, then the rhs of (3) is strictly positive, since double-oriented percolation is equivalent to oriented percolation in the first quadrant and simultaneous oriented percolation with inverted orientations in the third quadrant. Hence

$$p_c \geqslant 1 - (p^*)^{1/4}$$

The other bound,

$$p_c \leqslant 1 - p^*$$

is a trivial consequence of the fact that the model dominates the basic oriented model and that, as remarked before, the critical point of this latter model coincides with $1 - p^*$. ∎

Part (i) of the theorem applies to bootstrap percolation with $l = 2$, for which the result $p_c = 0$ had been proven by van Enter[3] based on an unpublished idea by Straley. The present proof was in fact inspired by these arguments, but in a sense it is more robust, since it applies to certain models in which the orientation is relevant and for which the Straley–van Enter argument does not apply. For instance, Duarte[2] considered the model for which a vacant site becomes occupied when at least two among its three neighbors to the north, east, and south are occupied. He observed that the Straley–van Enter argument does not apply and obtained from simulations on lattices of linear size between 5 and 15,360 the numerical prediction $p_c = 0.034 \pm 0.01$. For this model $\{\{N, E\}, \{S, E\}\} \subset \mathscr{D}$, so that the present theorem implies $p_c = 0$.

As a consequence of the present bounds on p_c, I conclude in particular that no model in the class considered can have p_c in the intervals $(0, 1 - (p^*)^{1/4})$ and $(1 - p^*, 1)$. The oriented models have $p_c = 1 - p^*$ and I conjecture that there are models that dominate an oriented model and have $0 < p_c < 1 - p^*$. A candidate for this seems to be the model defined by

$$\mathscr{D} = \{A \subset \mathscr{N}: \{N, E\} \subset A \text{ or } \{W, E\} \subset A\}$$

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